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ERROR PROBABILITIES FOR THE DETECTION OF RADAR TARGETS BY MISMATCHED RECEIVERS USING HALF-WAVE νth-LAW RECTIFIERS

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ABSTRACT

The probabilities of error and successful signal detection are determined for incoherent radar pulse trains observed by a suitably matched or mismatched receiver, which employs a half-wave νth-law rectifier before video presentation: single targets, normal noise, and incoherent reception are assumed. The effects of detector law, input signal strength, and sample size on performance are examined, and in particular, the modification of performance by mismatch is studied. Mismatch requires, not unexpectedly, a stronger input to achieve performance comparable to that at match. The amount and extent of these modifications are calculated for a variety of examples of interest. In all cases a (relatively) large-sample theory is required for the analysis when comparatively simple results are to be obtained. Exact expressions (all sample sizes) are found, however, in the quadratic case ($\nu = 2$) .

Accepted for the Air Force
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Error Probabilities for the Detection of Radar Targets by Mismatched Receivers Using Half-Wave νth-Law Rectifiers

I. INTRODUCTION

A moving target is observed in system noise. The error probabilities associated with the detection process are to be determined for a pulsed radar with incoherent detection (from pulse to pulse). The receiver itself consists of a suitably "matched" filter, which also takes the target's "gross" Doppler into account, and is followed by a general half-wave νth-law rectifier and video integrator. The "gross" Doppler implies envelope tracking but does not provide RF tracking, i.e., it is assumed that the reciprocal of the pulse length is much larger than the Doppler shift within each pulse. The gross Doppler is simultaneously measured and supplied to the detector for each received pulse. Because of "match," target amplitude remains essentially constant ($\equiv$ the further assumption here of no target scintillation) during an observation period. Furthermore, this period is usually short compared to the time it takes the target to move a distance great enough to affect the strength of the returned signal.

An undesired (i.e., clutter) target, also moving but at a different velocity to that of the desired target above, is alternatively detected in system noise. Because the receiver is matched to the Doppler of the desired target, it will <u>not</u> be matched to the undesired target, whose amplitude from pulse to pulse will now vary as the target "scans" through the matched filter in the given observation period. The central problem here is to calculate the error (and detection) probabilities for this "modulated," undesired target in system noise, and to compare detector performance with that of the desired or "matched" target above. In particular, we wish to determine quantitative relations between the error probabilities, the number of desired and undesired target returns, the input signal-to-noise ratio, and the law (ν) of the receiver's rectifying element for a variety of important operating conditions.

Detection takes place, accordingly, under the following circumstances:

(1) Independent noise (and signal) samples, equally spaced in the observation interval $(0, T)$. A one-point theory is assumed throughout;[1] i.e., a single sample value represents a noise, or noise and signal amplitude;

(2) The background noise is (white) normal, and stationary, with mean intensity $\psi = \overline{N^2}$, and zero mean $\overline{N} = 0$;

(3) Incoherent observation and waveforms: each pulse is independent of the others.

For the desired signal and filter matched to it, including the "gross" Doppler effect, the signal input to the rectifier is $S_d(t')$, where $t' = (1 + \epsilon_d)t$, and $1 + \epsilon_d$ is the change of scale produced by the relative target motion. Similarly, for the undesired (and mismatched) target's signal return, one has $S_c[(1 + \epsilon_c)t] = S_c[(1 + \epsilon_d + \epsilon_c - \epsilon_d)t] = S_c(t' + \epsilon_c' t')$, on the same fine scale, where $\epsilon_c' \equiv \epsilon_c - \epsilon_d (\neq 0)$.

If we suppose by way of illustration that the (returned) signals in both instances consist of periodic pulse trains made up of rectangular pulses, then the matched filter's output is a series of triangular pulses,* with periodic samples of fixed amplitude in the one-point theory, as shown in Fig. 1a. To a good approximation in the mismatched case, one still has a set of triangular pulses — now, however progressively shifted at the differential Doppler rate $(\sim \epsilon_c')$ with respect to the original matched filter's time scale, as shown in Fig. 1b. The "one-point" samples here are no longer of equal amplitude, but are "modulated" by the shifting triangular waveform of a typical output envelope of the matched filter, as indicated in the figure.

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* This is simply the correlation function of a typical pulse (envelope) when the background noise is white. See Ref. 1, for example; e.g., Eq. (16.106) in Eq. (16.96a) and pp. 132-133. See also Ref. 1.a, Sec. 4.2.

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1. D. Middleton, An Introduction to Statistical Communication Theory, (McGraw-Hill Co., New York, 1960), Sec. 20.4-3.

1a. __________, Topics in Communication Theory (McGraw-Hill Co., New York, 1965).

Note the amplitude modulation and the often sizeable periods when there is no target in the range (Doppler) gate in the mismatched case, when the gate is chosen for the desired target. Thus, the effect of mismatch here is to reduce the undesired (or clutter) target strength, both by modulation and by effectively removing the clutter return from a large number of the "matched" gate intervals. Accordingly, we may then ask how much stronger must a clutter target (of the same cross section as the desired target) be to give the same performance as the desired target: in other words, what is the clutter-to-signal (power) ratio for equivalent performance? In order to carry the analysis here to a quantitative conclusion, it is necessary to make two further simplifying assumptions in addition to that above of a "1-point" theory:[1]

(4) Replace the (essentially) triangular modulation by a simple rectangular one: Out of a total of n transmitted pulses, one has just $0 \leq m \ (< n)$ "hits" in the mismatched case. Each of these pulses to the receiver's second detector and integrator now has constant amplitude, A_{oc}. (In the matched case, $m = n$ and $A_{oc} = A_o$, the amplitude of a typical pulse.)

(5) Assume a large-sample theory (which is certainly required in the critical situation of weak desired signals). This means large n, 10-100 or more,(but not necessarily large m), and allows us to employ asymptotically normal statistics in the calculation of the desired error probabilities, since the noise components are statistically independent from sample to sample.

The early, classical study by Marcum[2] touched indirectly on the problem considered here (cf. "Collapsing Loss," pp. 213-215, Ref. 2). However, unlike this earlier work, the present study permits explicit results for the general νth-law case and emphasizes the specific relationships between the various controlling parameters involved in the present application. Section 2 below gives a general formulation for the calculation of the desired error probabilities, while Sec. 3 carries these relations through to specific results in the Gaussian approximation (cf. (5) above). Section 4 examines the weak-

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2. J. I. Marcum, " A Statistical Theory of Target Detection by Pulsed Radar," and "Mathematical Appendix," RAND, RM-754 (1 Dec. 1947); RM-753 (1 July 1948), see IRE Trans. on Info. Theory, Vol. IT-6 (April 1960), J. I. Marcum and P. Swerling.

and strong-signal situations and summarizes the principal analytic results, on which the curves and discussions of the concluding Sec. 5 are based.

II. GENERAL FORMULATION

Let us begin by developing the expressions for the error probabilities of target detection under the conditions of the model described above in Sec. 1. These error probabilities may be represented by*

$$\alpha = \int_K^{\infty} dz \int_{\Gamma} W_n (\underset{\sim}{E}|H_o)\delta (z - f(\underset{\sim}{E}))\, d\underset{\sim}{E} = \int_K^{\infty} q_n (z)\, dz \tag{2.1}$$

for the Type I error, or false-alarm probability, and by

$$\beta = \int_{-\infty}^{K} dz \int_{\Gamma} W_{n|m}(\underset{\sim}{E}|H_1)\, \delta (z - f(\underset{\sim}{E}))\, d\underset{\sim}{E} = \int_{-\infty}^{K} p_{n|m} (z)\, dz \tag{2.2}$$

for the Type II error, or false-dismissal probability. Thus, $1 - \beta$ is the conditional probability of successful target detection. Here we have

$\underset{\sim}{E} = (E_1, E_2, \ldots, E_n) =$ the vector whose components are the n pulse envelopes at the gated instants t_j $(j = 1, \ldots, n)$, which constitutes the n successive inputs to the rectifier and final low-pass filter (or integrator);

$g =$ detector law;

$z = f(E) = \sum_{j=1}^{n} g(E_j) = \gamma_\nu \sum_{j=1}^{n} E_j^{\nu}$, the output of the rectifier and integrator;

$K =$ decision threshold, at the output of the final integrator.

Here, H_o, H_1 signify the null (noise-alone) and signal hypotheses, respectively, while W_n, $W_{n|m}$ are the joint d.d.'s of $\underset{\sim}{E}$ when only noise is present and when m out of n pulses contain signal as well as noise. In addition, the d.d.'s of z are given by q_n and $p_{n|m}$, for which the associated

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* Ref. 1, Sec. 19.3-2, p. 815.

characteristic functions are:

$$F_1(i\xi|H_o) = \langle e^{i\xi f(\underset{\sim}{E})}\rangle_{H_o} \; ; \; F_1(i\xi|H_1) = \langle e^{i\xi f(\underset{\sim}{E})}\rangle_{H_{1:n|m}} \tag{2.3}$$

and

$$q_n(z) = \int_{-\infty}^{\infty} e^{-i\xi z} F_1(i\xi|H_o)\frac{dz}{2\pi} = p_{n|o}(z); \; p_{n|m}(z) = \int_{-\infty}^{\infty} e^{-i\xi z} F_1(i\xi|H_1)\frac{dz}{2\pi} \tag{2.4}$$

in the usual way.

For our present model, the required d.d.'s of the envelope are given by*

$$W_n(\underset{\sim}{E}|H_o) = \prod_{j=1}^{n} (E_j/\psi)e^{-E_j^2/2\psi} \quad , \; E_j \geq 0 \quad ; \tag{2.5}$$

$$W_n(\underset{\sim}{E}|H_1) = \prod_{j=1}^{n} (E_j/\psi)e^{-E_j^2/2\psi - A_o^2/2\psi} I_o(E_j A_o/\psi) \quad , \; E_j \geq 0 \quad . \tag{2.6}$$

In the general situation of m "hits" out of n received pulses, Eq. (2.5) applies in the null signal case, while for a signal we have

$$W_{n|m}(\underset{\sim}{E}|H_1) = W_m(\underset{\sim}{E}|H_1)\, W_{n-m}(\underset{\sim}{E}|H_o) \quad , \tag{2.7}$$

since the pulses are independent and the order of arrivals is thus not signifi cant. From Eq. (2.6) we get directly, with $p_{oc} \equiv A_{oc}{}^2/2\psi$,

$$W_{n|m}(E|H_1) = e^{-mp_{oc}} \prod_{j=1}^{m} (E_j/\psi)e^{-E_j^2/2\psi} I_o(A_{oc}E_j/\psi) \cdot \prod_{j=1}^{n} (E_j/\psi)e^{-E_j^2/2\psi} \quad E_j \geq 0 \; . \tag{2.8}$$

Inserting Eqs. (2.5) and (2.8) into (2.3) yields the characteristic functions needed for the calculation of the d.d.'s of the receiver's output, Eq. (2.4):

$$F_1(i\xi|H_o) = \left(\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}^2/2 + i\xi\gamma_\nu \psi^{\nu/2}\mathcal{E}^\nu} d\mathcal{E}\right)^n \tag{2.9}$$

and

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* Ref. 1, pp. 899-900.

and

$$F_1(i\xi|H_1) = e^{-mp_{oc}}\left(\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}^2/2+i\xi\gamma_\nu\psi^{\nu/2}\mathcal{E}^\nu} I_o(p_{oc}\sqrt{2}\mathcal{E})\, d\mathcal{E}\right)^m \cdot$$

$$\cdot\left(\int_0^\infty \mathcal{E}\, e^{-\mathcal{E}^2/2+i\xi\gamma_\nu\psi^{\nu/2}\mathcal{E}^\nu} d\mathcal{E}\right)^{n-m} \quad . \tag{2.10}$$

<u>The Quadratic Rectifier</u>

Closed-form expressions for $F_1(i\xi|H_o)$ are not possible, except in the important special case of the half-wave quadratic rectifier ($\nu = 2$) . For this case we find that

<u>($\nu = 2$):</u>

$$F_1(i\xi|H_o) = (1 - 2\psi\gamma_2\xi)^{-n} \quad , \tag{2.11a}$$

$$F_1(i\xi|H_1) = (1 - 2i\psi\gamma_2\xi)^{-n} e^{-mp_{oc}} \cdot e^{mp_{oc}(1-2\psi\gamma_2 i\xi)^{-1}} \quad , \tag{2.11b}$$

for which the Fourier transforms according to Eq. (2.4) may be shown to be,* after the substitution

$$z/2\psi\gamma_2 = y \quad , \tag{2.12}$$

respectively,

$$q_n(y)_{\nu=2} = \frac{y^{n-1}}{\Gamma(n)}\, e^{-y} \quad , \ y > 0 \ ; = 0 \ , \ y < 0 \ , \ (n \geq 1) \quad , \tag{2.13a}$$

and

$$p_n(y)_{\nu=2} = e^{-mp_{oc}}(y/mp_{oc})^{\frac{n-1}{2}}\, e^{-y} I_{n-1}(2\sqrt{mp_{oc}y}) \quad , \ y>0;=0, y<0 \ . \tag{2.13b}$$

(These are Marcum's relations, Eqs. (38), (49), Ref. 2, with appropriate modification of the parameters.) The desired error probabilities are from

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* With the help of No. 650.0, Campbell and Foster, "Fourier Integrals," (D. Van Nostrand, New York), 1954.

Eqs. (2.13) in (2.1, (2.2), $\nu = 2$:

$$\alpha_2 = 1 - I_c(K_o; n) \quad , \quad \text{with } K_o \equiv K/2\psi\gamma_2 \quad , \tag{2.14a}$$

and

$$\beta_2 = \int_0^{K_o} e^{-mp_{oc}-y} (y/mp_{oc})^{\frac{n-1}{2}} I_{n-1}(2\sqrt{mp_{oc}y}) \, dy \quad , \tag{2.14b}$$

where I_c is the incomplete Γ-function

$$I_c(x;n) \equiv \int_x^\infty z^{n-1} e^{-z} dz/\Gamma(n) \quad . \tag{2.14c}$$

The Type II error probability β_2 may be expressed as a special form of the incomplete Toronto function defined by*

$$T_A(p,q;r) \equiv 2r^{q-p+1} e^{-r^2} \int_0^A t^{p-q} e^{-t^2} I_q(2rt) \, dt \quad , \tag{2.15}$$

e.g.,

$$\beta_2 = T_{\sqrt{K_o}}(2n-1, n-1; \sqrt{mp_{oc}}) \quad , \quad 0 \le m \le n \quad . \tag{2.16}$$

Since I_c is a tabulated function,[3] and $T_{\sqrt{a}}(2n-1, n-1; \sqrt{b})$ has been tabulated by Marcum (Figs. 13-32, pp. 227-246, Ref. 2), we may say that exact results for the error probabilities α_2, β_2 are available in the quadratic case.

Frequently $\alpha (= \alpha_2)$ is preset, establishing the threshold K_o. Then, from Eq. (2.14a) we have

$$K_o = I_c^{-1}(1 - \alpha_2; n) \quad , \tag{2.17}$$

where I_c^{-1} is the inverse Γ-function, i.e., the function whose argument is

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* See pps. 167-170, and in particular, pp. 182-183 of Ref. 2.

3. K. Pearson, Tables of the Incomplete Γ-Function, (Cambridge University Press), 1951.

the Γ-function, so that the Type II error probability, Eq. (2.16), can be alternatively expressed as

$$\beta_2 = T_{\sqrt{I_c^{-1}(1-\alpha_2;n)}}(2n - 1, n - 1; \sqrt{mp_{oc}}) \quad . \tag{2.18}$$

Note, in particular, that <u>the Type II error probability for the quadratic case is a universal function of mp_{oc}</u>, the total received signal-to-noise (power) ratio.

However, as we can see from Eqs. (2.9, (2.10) in (2.4), etc., exact closed-form results are not possible for other detector laws ($\nu \neq 2$),* and we must have recourse to approximate methods, as described below in the large-sample situations. We observe, also from Eqs. (2.9), (2.10), that in general

$$\beta_\nu = F_n(m, p_{oc}) \quad , \text{ or } \hat{F}_n(m, mp_{oc}, p_{oc}) \quad , \quad 0 < \nu \quad , \tag{2.19}$$

namely, that the Type II error probability (and hence the probability of successful target detection) is <u>not</u> a universal function of mp_{oc} only, except when $\nu = 2$, cf. Eq. (2.18).

III. THE NORMAL APPROXIMATION: LARGE-SAMPLE CASES

The important case where the number (n) of pulses (signal and/or noise) is large can be handled, however, in a comparatively simple fashion, by taking advantage of the approach to normality of the distribution densities $W_n(\underset{\sim}{E}|H_o)$, $W_{n|m}(\underset{\sim}{E}|H)$, cf. Eqs. (2.5), (2.8). Our procedure here is to approximate the d.d.'s (2.5), (2.8) by a suitable Laguerre development, obtain the corresponding characteristic functions, and then expand these in an Edgeworth series. (This can be shown to be a somewhat briefer and more elegant equivalent to a direct expansion of Eqs. (2.9) and (2.10), with a suitable collection of terms and summation to provide the Edgeworth approximation.) The Laguerre (or similar "one-sided") development is, of course, required, since here the

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* Unless, of course, we are willing to define and tabulate a new hierarchy of special functions, even more involved than the Toronto functions above.

d. d.'s W_n, $W_{n|m}$ vanish for negative arguments. We have, in general,* after a change of variable

$$y = z/\gamma_2 \psi^{\nu/2} = \sum_{j=1}^{n} (E_j/\sqrt{\psi})^{\nu} \quad , \text{ cf. (2.1), (2.2) et seq.} \tag{3.1a}$$

$$p_{n|m}(y) = \sum_{k=0}^{\infty} A_k^{(n|m)} e^{-y} y^{\alpha} L_k^{(\alpha)}(y) \quad , \quad y \geq 0 \quad , \tag{3.2a}$$

where

$$A_0^{(n|m)} = \frac{1}{\beta\Gamma(\alpha+1)}$$

$$A_1^{(n|m)} = \frac{1}{\beta\Gamma(\alpha+2)} (1 + \alpha - \nu_1/\beta) \tag{3.2b}$$

$$A_2^{(n|m)} = \frac{1}{\beta\Gamma(\alpha+3)} [(\alpha+1)(\alpha+2) - \frac{2\nu_1}{\beta}(\alpha+2) + \frac{\nu_2}{\beta^2}] \quad , \text{ etc.} \tag{3.2b}$$

By choosing α and β so that $A_1^{(n|m)} = A_2^{(n|m)} = 0$, we find that* specifically

$$A_0^{(n|m)} = \frac{\nu_1}{\sigma^2\Gamma(\nu_1^2/\sigma^2)} \quad ; \quad A_3^{(n|m)} = \frac{1}{\beta\Gamma(\alpha+4)} \left[\frac{\nu_2}{\beta^2}(\alpha+3) - \frac{\nu_3}{\beta^3}\right] \quad , \tag{3.3}$$

where

$$\alpha = \nu_1^2/\sigma^2 - 1 \quad ; \quad \beta = \sigma^2/\nu_1 \quad . \tag{3.3a}$$

Here ν_1 is the first-moment of y, and σ^2 is the variance of y, with respect to the d. d. $p_{n|m}(y)$. These quantities depend, of course, on n and m: thus, we have, since

$$p_{n|0}(y) = q_n(y) \quad , \tag{3.4}$$

explicitly ,

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* Ref. 2, pp. 189-192.

$$\left.\begin{aligned}
&\nu_{1|o} \equiv \overline{y}^{H_o} = \int y q_n(y)\, dy \quad ; \quad \nu_{1|1} \equiv \overline{y}^{H_1} = \int y\, p_{n|m}(y)\, dy \quad ; \text{ etc.,} \\
&\text{and} \\
&\sigma_o^2 \equiv \overline{y^2}^{H_o} - (\overline{y}^{H_o})^2 \quad ; \quad \sigma_1^2 \equiv \overline{y^2}^{H_1} - (\overline{y}^{H_1})^2 \quad , \\
&\text{with} \\
&\overline{y^k}^{H_o} \equiv \nu_{k|o} = \int y^k q_n(y)\, dy \quad ; \quad \overline{y^k}^{H_1} \equiv \nu_{k|1} = \int y^k p_{n|m}(y)\, dy \quad .
\end{aligned}\right\} \tag{3.5}$$

Only the leading term is used in this approximation, to give us

$$p_{n|m}(y) \doteq \frac{\nu_{1|1}}{\sigma_1^2 \Gamma(\nu_{1|1}^2/\sigma_1^2)}\, e^{-\nu_{1|1} y/\sigma_1^2} \left(\frac{\nu_{1|1} y}{\sigma_1^2}\right)^{\nu_{1|1}^2/\sigma_1^2 - 1} \quad , \quad y \geq 0 \tag{3.6}$$

and

$$p_{n|o}(y) = q_n(y) \doteq \frac{\nu_{1|o}}{\sigma_o^2 \Gamma(\nu_{1|o}^2/\sigma_o^2)}\, e^{-\nu_{1|o} y/\sigma_o^2} \left(\frac{\nu_{1|o} y}{\sigma_o^2}\right)^{\nu_{1|o}^2/\sigma_o^2 - 1} \quad , \quad y \geq 0 \, . \tag{3.7}$$

The d.d.'s vanish, of course, for $y < 0$.

In terms of the detector characteristic studied here (cf. Eq. (2.2) et seq.), we can now write in more detail the various moments

$$(\nu_{1|1})_m = \overline{y_{(m)}}^{H_1} = \sum_{j=1}^{m} \overline{g(E_j)}^{H_1} = m\, \overline{g(E)}^{H_1} \equiv m\, \hat{\nu}_{1|1} \tag{3.8a}$$

$$(\nu_{1|o})_n = \overline{y_{(n)}}^{H_o} = \sum_{j=1}^{n} \overline{g(E_j)}^{H_o} = n\, \overline{g(E)}^{H_o} \equiv n\, \hat{\nu}_{1|o} \tag{3.8b}$$

where

$$\hat{\nu}_{1|1} \equiv \overline{g(E)}^{H_1} = \overline{y_{(1)}}^{H_1} \quad ; \quad \hat{\nu}_{1|o} \equiv \overline{g(E)}^{H_o} = \overline{y_{(1)}}^{H_o} \tag{3.8c}$$

Similarly, we have

$$(\sigma_1^2)_m = \overline{y_{(m)}^2}^{H_1} - \left(\overline{y_{(m)}}^{H_o}\right)^2 = m\left[\overline{y_{(1)}^2}^{H_1} - \left(\overline{y_{(1)}}^{H_1}\right)^2\right] \equiv m\hat{\sigma}_1^2 \text{ ,etc.} \tag{3.9a}$$

with

$$\hat{\sigma}_1^{\,2} \equiv \overline{g(E)^2}^{H_1} - \left(\overline{g(E)}^{H_1}\right)^2 = \overline{y_{(1)}^2}^{H_1} - \left(\overline{y_{(1)}}^{H_1}\right)^2 \quad , \text{ etc.} \tag{3.9b}$$

Let us now introduce

$$\lambda_o \equiv \hat{\nu}_{1|o}^{\,2}/\hat{\sigma}_o^{\,2} \; ; \; \lambda_1 \equiv \hat{\nu}_{1|1}^{\,2}/\hat{\sigma}_1^{\,2} \; ; \; \beta_o \equiv \hat{\sigma}_o^{\,2}/\hat{\nu}_{1|o} \; ; \; \beta_1 \equiv \hat{\sigma}_1^{\,2}/\hat{\nu}_{1|1} \; . \tag{3.10}$$

Then, taking the Fourier transforms of Eqs. (3.6) and (3.7) and using Eqs. (3.8)-(3.10), we find that the corresponding characteristic functions are

$$F_1(i\xi|H_1) = [1 - i\xi\beta_o]^{-(n-m)\lambda_o}\,[1 - i\xi\beta_1]^{-m\lambda_1} \tag{3.11}$$

$$F_1(i\xi|H_o) = [1 - i\xi\beta_o]^{-n\lambda_o} \quad . \tag{3.12}$$

Our next step is to expand $F_1(i\xi|H_1)$:

$$F_1(i\xi|H_1) = \exp\left[-(n-m)\lambda_o \log(1 - i\xi\beta_o) - m\lambda_1 \log(1 - i\xi\beta_1)\right] \tag{3.13a}$$

$$= \exp\left(a_1^{(1)} i\xi - \frac{a_2^{(1)}\xi^2}{2} - \frac{i\xi^3}{3} a_3^{(1)} + \dots\right) \quad , \tag{3.13b}$$

where specifically

$$\left.\begin{aligned} a_1^{(1)} &= (n - m)\,\lambda_o\beta_o + m\lambda_1\beta_1 \\ a_2^{(1)} &= (n - m)\,\lambda_o\beta_o^{\,2} + m\lambda_1\beta_1^{\,2} \\ a_3^{(1)} &= (n - m)\,\lambda_o\beta_o^{\,3} + m\lambda_1\beta_1^{\,3} \quad , \text{ etc.} \end{aligned}\right\} \tag{3.14}$$

The Edgeworth expansion* for the characteristic function is accordingly

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* Ref. 1, pp. 366-367, Problem (7.11).

$$F(i\xi|H_1) \doteq e^{i\xi a_1^{(1)} - a_2^{(1)}\xi^2/2}\left[1 + \frac{a_3^{(1)}}{3}(i\xi)^3 + \dots\right] \quad , \tag{3.15a}$$

and the corresponding d.d. is

$$p_{n|m}(y) \simeq \frac{1}{\sqrt{2\pi a_2^{(1)}}}\left[e^{-(y-a_1^{(1)})^2/2a_2^{(1)}} - \frac{\sqrt{2\pi}a_3^{(1)}}{3(a_2^{(1)})^{3/2}}\phi^{(3)}(y_1) + \dots\right], \tag{3.15b}$$

with

$$\left.\begin{aligned} y_1 &= [y - a_1^{(1)}]/\sqrt{a_2^{(1)}} \\ \phi^{(3)}(y_1) &= \frac{d^3}{dy_1^3}\frac{e^{-y_1^2/2}}{\sqrt{2\pi}} \end{aligned}\right\} \quad .$$

A similar calculation shows that

$$q_n(y) \simeq \frac{1}{\sqrt{2\pi a_2^{(0)}}}\left[e^{-(y-a_1^{(0)})^2/2a_2} - \frac{\sqrt{2\pi}\, a_3^{(0)}}{3[a_2^{(0)}]^{3/2}}\phi^{(3)}(y_o) + \dots\right] , \tag{3.16}$$

with

$$y_o \equiv [y - a_1^{(0)}]/\sqrt{a_2^{(0)}} \quad ,$$

where now

$$a_1^{(0)} = n\lambda_o\beta_o \;;\; a_2^{(0)} = n\lambda_o\beta_o^2 \;;\; a_3^{(0)} = n\lambda_o\beta_o^3 \quad , \quad \text{etc.} \tag{3.17}$$

Inserting Eqs. (3.15b), (3.16) into (2.2), (2.1) allows us to write finally for the desired error probabilities

$$a_\nu \simeq \frac{1}{2}\left\{1 - H\left[\frac{K_o - a_1^{(0)}}{\sqrt{2a_2^{(0)}}}\right]\right\} + \frac{a_3^{(0)}}{3[a_2^{(0)}]^{3/2}}\phi^{(2)}\left[\frac{K_o - a_1^{(0)}}{\sqrt{a_2^{(0)}}}\right] + \dots \tag{3.18a}$$

$$\beta_\nu \simeq \frac{1}{2}\left\{H\left[\frac{K_o - a_1^{(1)}}{\sqrt{2a_2^{(1)}}}\right] + H\left[\frac{a_1^{(1)}}{\sqrt{2a_2^{(1)}}}\right]\right\} - \frac{a_3^{(1)}}{3[a_2^{(1)}]^{3/2}}\phi^{(2)}\left[\frac{K_o - a_1^{(1)}}{\sqrt{a_2^{(1)}}}\right] + \dots, \tag{3.18b}$$

where now the threshold K_o is given by

$$K_o = K/\gamma_\nu \beta_o \psi^{\nu/2} \quad , \quad \text{and } \textcircled{H}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{3.18c}$$

is the familiar error integral.* When $a_3/3a_2^{3/2}$ is sufficiently small (the "large-sample" case, n large), we can drop the correction terms in Eq. (3.18a, b). Moreover, it is then a simple matter to eliminate the threshold K_o and express the type II error probability β_ν in terms of the false-alarm probability, α_ν, by

$$\beta_\nu \simeq \frac{1}{2}\left\{ \textcircled{H}\left[\frac{a_1^{(1)}}{\sqrt{2a_2^{(1)}}}\right] - \textcircled{H}\left[\frac{a_1^{(1)} - a_1^{(0)} - \sqrt{2a_2^{(0)}}\,\textcircled{H}^{-1}(1-2\alpha_\nu)}{\sqrt{2a_2^{(1)}}}\right]\right\}, \; n>>1 \quad . \tag{3.19}$$

For the most part, this is the general expression which we shall use in later sections and from which the various results of the figures are specifically obtained, for various values of detector law, ν.

IV. MOMENTS: WEAK- AND STRONG-SIGNAL APPROXIMATIONS

To evaluate the error probability β_ν, we must specify the parameters of Eq. (3.19) with the help of Eqs. (3.8c), (3.9b), (3.10), (3.14), and (3.17). First, we have from Eq. (9.52) of Ref. 1

$$\overline{y_{(1)}}^{H_1} = \hat{\nu}_{1|1} = 2^{\nu/2}\Gamma(\nu/2+1)\,{}_1F_1(-\nu/2; 1; -p_{oc}) \quad , \tag{4.1}$$

$$\therefore \hat{\nu}_{1|o} = 2^{\nu/2}\Gamma(\nu/2+1) \; (= \hat{\nu}_{1|1} \text{ with } p_{oc} = 0) \quad , \tag{4.1a}$$

and

$$\overline{y_{(1)}^2}^{H_1} = 2^\nu \Gamma(\nu+1)\,{}_1F_1(-\nu; 1; -p_{oc}) \quad , \tag{4.2}$$

$$\therefore \overline{y_{(1)}^2}^{H_o} = 2^\nu \Gamma(\nu+1) \quad . \tag{4.2a}$$

---

* Appendix A-1.1 of Ref. 1, and Ref. 3 therein.

Here ${}_1F_1$ is the familiar confluent hypergeometric function, some of whose principal properties are discussed in Appendix A-1.2, Ref. 1. For useful tables, see Ref. 9 therein.

Consequently, we have

$$a_1^{(1)} = (n-m)\hat{\nu}_{1|o} + m\hat{\nu}_{1|1} = 2^{\nu/2}\Gamma(\nu/2+1)\{n-m+m\,{}_1F_1(-\nu/2;1;-p_{oc})\}\ , \quad (4.3)$$

$$\therefore a_1^{(0)} \qquad = 2^{\nu/2}\Gamma(\nu/2+1)\cdot n \quad . \qquad (4.3a)$$

Similarly, we get

$$a_2^{(1)} = (n-m)\,\hat{\sigma}_o^{\,2} + m\hat{\sigma}_1^{\,2} = 2^{\nu}\,\Gamma\,(\nu/2+1)^2$$

$$\cdot \left[(n-m)\gamma + m\left\{\frac{\Gamma\,(\gamma+1)}{\Gamma(\nu/2+1)^2}\ {}_1F_1(-\nu;1;-p_{oc}) - {}_1F_1^{\,2}(-\nu/2;1;-p_{oc})\right\}\right] \ , \quad (4.4)$$

$$\therefore a_2^{(0)} \qquad = 2^{\nu}\,\Gamma/\nu/2+1)^2\,n\gamma \quad , \qquad (4.4a)$$

where

$$\gamma \equiv \Gamma\,(\nu+1)/\Gamma\,(\nu/2+1)^2 - 1\ (>0) \quad , \ \nu > 0 \quad . \qquad (4.5)$$

(For the correction term we need

$$a_3^{(1)} = (n-m)\,\hat{\sigma}_o^{\,4}/\hat{\nu}_{1|o} + \hat{\sigma}_1^{\,4}/\hat{\nu}_{1|1} \quad . \qquad (4.6)$$

Noting that

$$\hat{\sigma}_1^{\,2} = 2^{\nu}\Gamma(\nu+1)\,{}_1F_1(-\nu;1;-p_{oc}) - 2^{\nu}\Gamma(\nu/2+1)^2\,{}_1F_1^{\,2}(-\nu/2;1;-p_{oc}) \qquad (4.7a)$$

$$\hat{\sigma}_o^{\,2} = 2^{\nu}\Gamma(\nu/2+1)^2\gamma \quad , \qquad (4.7b)$$

we see that

$$\frac{a_3^{(1)}}{3[a_2^{(1)}]^{3/2}} = \frac{(n-m)\gamma^2 + m[\frac{\Gamma\,(\nu+1)}{\Gamma(\nu/2+1)^2}\,{}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)]^2/{}_1F_1(-\nu/2)}{3\left\{(n-m)\gamma + m[\frac{\Gamma/\nu+1)}{\Gamma(\nu/2+1)^2}\ {}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)]\right\}^{3/2}} \ , \quad (4.8)$$

where we abbreviate ${}_1F_1(-\nu) \equiv {}_1F_1(-\nu;1;-p_{oc})$, etc., and that

$$\frac{a_3^{(0)}}{3[a_2^{(0)}]^{3/2}} = \frac{1}{3}\sqrt{\frac{\gamma}{n}} \tag{4.9}$$

for the case of noise alone.)

Substitution of Eqs. (4.3)-(4.4a) into Eq. (3.19) yields the following general, large-sample result:

$$\beta_\nu \simeq \frac{1}{2}\left\{\textcircled{H}\left[\frac{(n-m) + m\,{}_1F_1(-\nu/2)}{\sqrt{2}\left[(n-m)\gamma + m\left\{\frac{\Gamma(\nu+1)}{\Gamma(\nu/\beta+1)^2}\,{}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)\right\}\right]^{1/2}}\right]\right.$$

$$\left. - \textcircled{H}\left[\frac{m[{}_1F_1(-\nu/2) - 1] - \sqrt{2n\gamma}\,\textcircled{H}^{-1}(1-2\alpha_\nu)}{\sqrt{2}\left[(n-m)\gamma + m\left\{\frac{\Gamma(\nu+1)}{\Gamma(\nu/2+1)^2}\,{}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)\right\}\right]^{1/2}}\right]\right\},\ n\gg 1, \tag{4.10}$$

which holds for all input signal-to-noise ratios $p_{oc} \geq 0$. Note that when a properly matched filter is used, $m = n$ and Eq. (4.10) becomes

$$\beta_{\nu|m=n} \simeq \frac{1}{2}\left\{\textcircled{H}\left[\frac{\sqrt{n}\,{}_1F_1(-\nu/2)}{\sqrt{2}\left[\frac{\Gamma(\nu+1)}{\Gamma(\nu/2+1)^2}\,{}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)\right]^{1/2}}\right]\right.$$

$$\left. - \textcircled{H}\left[\frac{\sqrt{n}[{}_1F_1(-\nu/2) - 1] - \sqrt{2\gamma}\,\textcircled{H}^{-1}(1-2\alpha_\nu)}{\sqrt{2}\left[\frac{\Gamma(\nu+1)}{\Gamma(\nu/2+1)^2}\,{}_1F_1(-\nu) - {}_1F_1^{\,2}(-\nu/2)\right]^{1/2}}\right]\right\},\ p_{oc} = p_o, \tag{4.11}$$

while if the predetector filter is completely mismatched to an undesired target, $m = 0$ and Eq. (4.10) reduces to

$$\beta_{\nu|m=o} \simeq \frac{1}{2}\left\{\textcircled{H}\left(\sqrt{\frac{n}{2\gamma}}\right) + 1 - 2\alpha_\nu\right\}, \tag{4.12}$$

which for sufficiently large samples becomes $\beta_\nu = 1 - \alpha_\nu$, or $\alpha_\nu + \beta_\nu = 1$, as is the case for $m \neq 0$ also.

Special cases of interest occur when the input signal is weak, $p_{oc}^2 << 1$, or strong, $(p_{oc} >> 1)$. Since

$$
\left.
\begin{aligned}
{}_1F_1(-\nu/2;1;-p_{oc}) &\doteq 1 + \frac{\nu}{2} p_{oc} + 0\,(p_{oc}^2) \quad , \quad p_{oc}^2 << 1 \quad , && (4.13a)\\
{}_1F_1(-\nu/2;1;-p_{oc}) &\simeq \frac{p_{oc}^{\nu/2}}{\Gamma(\nu/2+1)} \left\{1 + \frac{\nu^2}{4\,p_{oc}} + \ldots\right\} \quad , \quad p_{oc}^2 >> 1 \quad , && (4.13b)\\
\text{and} \qquad & \\
{}_1F_1(-\nu;1;-p_{oc}) &\doteq 1 + \nu\, p_{oc} + 0\,(p_{oc}^2) \quad , \quad p_{oc}^2 << 1 \quad , && (4.14a)\\
&\simeq \frac{p_{oc}}{\Gamma(\nu+1)} \left\{1 + \frac{\nu^2}{p_{oc}^2} + \ldots\right\} \quad , \quad p_{oc}^2 >> 1 \quad , && (4.14b)
\end{aligned}
\right\}
$$

we obtain for Eq. (4.10)

weak inputs:

$$
\beta_\nu \simeq \frac{1}{2}\left\{ \Theta\left[\frac{n + \frac{m\nu}{2} p_{oc}}{\sqrt{2(n+\nu m p_{oc})\gamma}}\right] - \Theta\left[\frac{\frac{m\nu}{2} p_{oc} - \sqrt{2n\gamma}\,\Theta^{-1}(1-2a_\nu)}{\sqrt{2(n+\nu m\, p_{oc})\,\gamma}}\right]\right\} \quad , \quad p_{oc}^2 << 1 \quad , \qquad (4.15)
$$

and

strong inputs:

$$
\beta_\nu \simeq \frac{1}{2}\left\{ \Theta\left[\frac{(n-m) + m\, p_{oc}^{\nu/2}/\Gamma(\nu/2+1)}{\{2(n-m)\gamma + m p_{oc}^{\nu-1}\nu^2/\Gamma(\nu/2+1)^2\}^{1/2}}\right] - \Theta\left[\frac{m p_{oc}^{\nu/2}/\Gamma(\nu/2+1) - \sqrt{2n\gamma}\,\Theta^{-1}(1-2a_\nu)}{\{2(n-m)\gamma + m\, p_{oc}^{\nu-1}\nu^2/\Gamma(\nu/2+1)^2\}^{1/2}}\right]\right\}, \quad p_{oc}^2 >> 1. \qquad (4.16)
$$

Certain general observations can be made:

(i) In weak-signal performance: $\beta_\nu = \beta_\nu(m\nu\, p_{oc};n;\nu)$;

(ii) In strong-signal operation: $\beta_\nu = \beta_\nu(m\, p_{oc}^{\nu/2};n;\nu)$, provided $m\,p_{oc}^{\nu-1}\nu^2/\Gamma(\nu/2+1)^2 << 2\,(n-m)\,\gamma$;

(iii) At intermediate input signal levels, β_ν is no longer a universal function of $m\nu\, p_{oc}$, or $m\, p_{oc}{}^{\nu/2}$. Thus, (i) and (ii) indicate canonical behavior, while (iii) shows its breakdown, with respect to the number of signal pulses integrated and the input signal-to-noise ratio, p_{oc}.

Special cases of interest occur for $\nu = 1$, $\nu = 2$. We have for the half-wave linear rectifier ($\nu = 1$):

$$\beta_1 \simeq \left\{ \Theta\left[\frac{(n-m) + m\, {}_1F_1(-1/2;\, 1;\, -p_{oc})}{[(n-m)(\frac{8}{\pi}-2) + m\{\frac{8}{\pi}(1+p_{oc}) - 2\, {}_1F_1{}^2(-1/2;1;\, -p_{oc})\}]^{1/2}} \right] \right.$$

$$\left. - \Theta\left[\frac{m\{{}_1F_1(-1/2;1;\, -p_{oc}) - 1\} - \sqrt{(\frac{8}{\pi} - 2)n}\, \Theta^{-1}(1-2a_1)}{[(n-m)(\frac{8}{\pi} - 2) + m\{\frac{8}{\pi}(1+p_{oc}) - 2\, {}_1F_1{}^2(-1/2;\, 1;\, -p_{oc})\}]^{1/2}} \right] \right\}, \quad (4.17)$$

with $\gamma\big|_{\nu=1} = \frac{4}{\pi} - 1$. The weak- and strong-signal versions of this are

weak-signal:

$$\beta_1 \simeq \frac{1}{2}\left\{ \Theta\left[\sqrt{\frac{\pi}{2(4-\pi)}}\, \frac{n+m\, p_{oc}/2}{\sqrt{n+m\, p_{oc}}} \right] \right.$$

$$\left. - \Theta\left[\sqrt{\frac{\pi}{2(4-\pi)}} \cdot \frac{\frac{m\, p_{oc}}{2} - \sqrt{n}\sqrt{(\frac{8}{\pi} - 2)}\, \Theta^{-1}(1 - 2a_1)}{\sqrt{n + m\, p_{oc}}} \right] \right\} \quad (4.18)$$

strong-signal:

$$\beta_1 \simeq \frac{1}{2}\left\{ \Theta\left[\frac{(n-m) + m\sqrt{p_{oc}} \cdot 2/\sqrt{\pi}}{\sqrt{(n-m)(\frac{8}{\pi} - 2) + \frac{4m}{\pi}}} \right] \right.$$

$$\left. - \Theta\left[\frac{\frac{2m}{\sqrt{\pi}}\sqrt{p_{oc}} - \sqrt{n(\frac{8}{\pi} - 2)}\, \Theta^{-1}(1 - 2a_1)}{\sqrt{(n-m)(\frac{8}{\pi} - 2) + \frac{4m}{\pi}}} \right] \right\}. \quad (4.19)$$

For the half-wave quadratic rectifier ($\nu = 2$), we get similarly, now <u>for all signal levels</u>:

$$\beta_2 \simeq \frac{1}{2}\left\{ \Theta\left[\frac{n+m\, p_{oc}}{\sqrt{2}\,\sqrt{n+2m\, p_{oc}}}\right] - \Theta\left[\frac{m\, p_o - \sqrt{2n}\,\Theta^{-1}(1-2a_2)}{\sqrt{2}\,\sqrt{n+2m\, p_{oc}}}\right]\right\} , \qquad (4.20)$$

where $\gamma \Big|_{\nu=2} = 1$.

When the mismatch is strong ($n >> m$) <u>and</u> $\nu \le 2$, we can simplify Eqs. (4.15), (4.16):

<u>weak inputs</u>:

$$\beta_\nu \simeq \frac{1}{2}\left\{ \Theta\left[\frac{n + \frac{m\nu}{2} p_{oc}}{\sqrt{2n\gamma}}\right] - \Theta\left[\frac{-\frac{m\nu}{2} p_{oc}}{\sqrt{2n\gamma}} - \Theta^{-1}(1-2a_\nu)\right]\right\} , \qquad (4.21)$$

$$n >> \nu m\, p_{oc};\ p_{oc}^2 << 1;\ \nu \le 2;\ (n >> m) , \qquad (4.21a)$$

<u>strong inputs</u>:

$$\beta_\nu \simeq \frac{1}{2}\left\{ \Theta\left[\frac{n + m\, p_{oc}^{\nu/2}/\Gamma(\nu/2+1)}{\sqrt{2n\gamma}}\right] - \Theta\left[\frac{m\, p_{oc}^{\nu/2}/\Gamma(\nu/2+1)}{\sqrt{2n\gamma}} - \Theta^{-1}(1-2a_\nu)\right]\right\} , \qquad (4.22)$$

$$n >> \frac{m\nu^2 p_{oc}^{\nu-1}}{2\gamma\, \Gamma(\nu/2+1)^2} ,\quad p_{oc}^2 >> 1;\ n >> m . \qquad (4.22a)$$

In particular, we have for the conditions governing Eqs. (4.21), (4.22):

$\nu = 2$: (all signals): $n >> 2m\, p_{oc}$, $p_{oc}^2 << 1;\ n >> m$

$\nu = 1$: (weak-signals): $n >> m\, p_{oc} (< m)$, " ; "

(strong-signals: $n >> n/(2 - \pi/2) \approx 2m$; (4.23)

$\nu = \frac{1}{2}$: (weak-signals): $n >> \frac{1}{2} m\, p_{oc}$

(strong-signals): $n >> m\, p_{oc}^{-1/2}/8[\Gamma(3/2) - \Gamma(5/4)^2]$, etc.

We also remark that when $\nu \to 0$, $\beta_\nu \to 1 - \alpha_\nu$: as "super-clipping" sets in, the fluctuations vis-à-vis the mean approach infinity, and it is no longer possible to distinguish the presence of signal, which for other rectifier characteristics ($\nu > 0$) manifests itself as a shift in the means of the distribution vis-à-vis a finite background variance [cf. Eq. (3.19)]. In fact, it can be shown that

$$\beta_{|\nu\to o, m=n} \simeq 1 - \alpha_\nu - \frac{\nu p_{oc}}{2} \sqrt{\frac{n}{2\pi\gamma}}\, e^{2\alpha - 1} + 0(\nu); \gamma = 0(\nu) \quad , \tag{4.24a}$$

$$\therefore \beta \simeq 1 - \alpha_{\nu = o} \quad . \tag{4.24b}$$

Correction terms have also been determined, cf. Eqs. (4.8) and (4.9) in (3.18), although for most applications here, n is sufficiently large (10-100 or more) to obviate their need. For example, if we are willing to tolerate a shift in threshold (K_o) of 10% or less (for $\alpha = 10^{-5}$), this produces a consequently negligible change in β_ν of $0(10^{-3})$ for the half-wave quadratic detector ($\nu = 2$), ($n = 10$, $p_o = 1$, $m = 1$) , with similar, ignorable changes in β_ν (or $= - \beta_\nu$, the detection probability) for other ν, m . As n increases, these effects become progressively diminished, as is also true for larger m and p_o .

V. RESULTS AND CONCLUSIONS

In the preceding sections we have determined the error probabilities for the detection of an incoherent pulse train of radar signals, observed both when the receiver is matched to the desired, or true, moving target (and when mismatch occurs, for undesired or false targets). A "matched" filter, followed by a half-wave νth-law rectifier, constitutes the critical receiver elements, with, of course, the usual RF and video components. In particular, we are interested in the probability of successful signal detection, $P_D = 1 - \beta_\nu$, when the false-alarm probability α_ν is preset at some appropriate value. The conditions under which our results apply are summarized as follows:

(i) n = total number of transmitted pulses, ≥ 10, for the "large-sample" approximations employed in the preceding analysis;

(ii) m = number of received signal pulses; $0 \leq m < n$ represents "mismatch," for an undesired target; $m = n$ represents "match," or the desired target;

(iii) A_{oc} = peak amplitude of the target return when $m < n$; ($A_{oc} = A_o$ for $m = n$, with A_o = amplitude of desired target return.);

(iv) p_{oc} = input signal-to-noise (power) ratio (on per-pulse basis) in the general case $1 \leq m < n$; ($p_{oc} = p_o$, when $m = n$).

Additive normal background noise is assumed, as usual, but the input signal level is arbitrary, ranging from threshold operation (p_o^2, $p_{oc}^2 << 1$) to the strong-signal cases (p_o^2, $p_{oc}^2 >> 1$), and the detector law ν ranges from (nearly) zero to 2, or more. For the most part, we are concerned here with values of ν in the interval ($0.1 \leq \nu \leq 2$). Similarly, n ranges from 10 to 10^3, and with $m \leq n$.

In a general way we wish to examine the effects on system performance, as measured by the probability of (correct) signal detection $P_D = 1 - \beta_\nu$ when α_ν is specified, of:

1. Sample size (n) and "mismatch" ($m < n$);
2. Input signal level ($n\, p_{oc}$, p_o);
3. Detector law, ν.

Selected values of the parameters are chosen for the curves of Figs. 2-11, which represent typical regions of interest. For other regions, we may use the analytic results of Secs. 2 through 4 in a similar fashion.

Four principal sets of situations may be individually compared: these are:*

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* We recall, of course, that only one target, desired or undesired, is assumed to be present in a given observation period in the present study. The extension to the case of two (or more) simultaneous targets is reserved to a possible later study.

1. Matched, small-signal (MS) vs mismatched, small-signal ($\overline{MS}$);

2. Matched, large-signal (ML) vs mismatched, small-signal ($\overline{MS}$);

3. Matched, large-signal (ML) vs mismatched, large-signal ($\overline{ML}$);

4. Matched, small-signal (MS) vs mismatched, large-signal ($\overline{ML}$).

Of these, (4) is particularly important when we wish to determine the effects of strong interference against a weak, desired target, while (2) may usually be ignored, as the system is then functioning adequately: the undesired signals ($\overline{MS}$) are quite outweighed by the desired ones, (ML). The other cases, (1) and (3), are of general interest.

Let us consider the figures:* Figure 2 gives the probability of detection, P_D, for the matched case ($m = n$) and weak inputs, for $\nu = 0.2,\ 1.0$, when sample size is variable. Three input signal levels are assumed: p_o ($= p_{oc}$) = -10 db, -5.0 db, 0.0 db. These curves are calculated from Eq. (4.15), or (4.18) for $\nu = 1$, except when $p_o = 0.0$ db, where the "exact" relation (4.11) must be used. (The results for $\nu = 2$ are not shown but are essentially the same as for $\nu = 1$.) The behavior of P_D is as expected: larger samples lead to larger values of P_D; smaller input signals require more pulses, and decreasing the detector law below $\nu = 1$ produces a degradation in performance because of the trend toward limiting that arises when $\nu < 1$. Note that the "loss" for mismatch — i.e., the <u>increase</u> in signal strength (over p_o) required to produce the same detection probability for given n, ν, may be determined here from the simple relation

$$p_{oc} = \frac{n}{m}\, p_o \quad , \quad (\text{all } \nu) \tag{5.1}$$

which follows at once from Eq. (4.15) and the dependence of β_ν, and $\therefore$ P_D, on the universal function $m\nu\, p_{oc}$ for this weak-signal régime.

Figure 3 provides a check on the weak- and strong-signal approximations (Eqs. (4.15) and (4.16)), to show how large p_o (~ 1.0) can be taken and still allow us to use the simpler, approximate relations, rather than the exact form (4.17), for $\nu = 1$ here. We note that for large inputs ($p_o \sim 1$) and $n > 40$, $P_D \geq 0.95$.

---

* In all cases, unless otherwise indicated, a false-alarm probability of $\alpha_\nu = 10^{-6}$ is postulated here.

Again, the matched case $(m = n)$ is assumed. Also, the small-signal approximation is seen to be conservative in that it underestimates P_D, while the large-signal approximation is somewhat under-conservative, giving too large a value of P_D, which, however, is not critical here, as we have noted above. The exact form lies between the two approximations.

Figure 4 repeats Fig. 3, but now for $\nu = 0.2$, with essentially similar results. Again, the matched case is assumed. We note, however, that the small signal-to-noise approximation is now under-conservative (too large values of P_D), while the strong-signal approximation is grossly under-conservative.*

Figure 5 shows the dependence of

$$K_s(\nu) = p_o(\nu/\beta = 1/2)\sqrt{n} \tag{5.2}$$

on ν, obtained from Eq. (4.15) on setting $\beta_\nu = P_D = 1/2$ in the matched, weak-signal régime, where $m\ (= n)$ is large enough to permit us to set the first error integral in Eq. (4.15) effectively equal to unity, the argument of the second equal to zero, and solve for $p_o(\nu)$ as ν is varied, β_ν, α_ν held fixed. When P_D is in the vicinity of one-half, the result indicates a relatively weak dependence of K_s on ν. Thus, we may say to a good degree of accuracy, for weak signals and practical detector laws ($\nu > 0$), that the input signal-to-noise ratio varies as the inverse square-root of the sample size, as noted in earlier work[2] in the special cases of the linear and quadratic rectifiers.

Figure 6 is based on Eq. (4.15) again and shows in this weak-signal règime the penalty ($\sim m/n$) [cf. Eq. (5.1)] incurred by mismatch, i.e., by how much the undesired target return must be greater than that of a desired one in order to yield the same detection probability. Of course, as m increases (to n), this penalty decreases proportionally. This penalty is here independent of detector law, ν.

Now consider the strong-signal situation (large samples) and mismatch, based on Eq. (4.16). Here we have

---

* We can estimate with the help of the curve for $P_D = 5$ db that the approximate large-signal formula for P_D here gives a value of n which is an order of magnitude too small.

$$p_{oc}(\nu) = [K_L(\nu)\, n/m^2]^{1/\nu}; \quad K_L(\nu) = 2\,(3.29)^2[\Gamma(\nu+1) - \Gamma(\nu/2+1)^2] \qquad (5.3)$$

with $P_D = 1/2$ (and the argument of the first error integral in Eq. (4.16) greater than 2.5), obtained by setting the numerator of the second error function equal to zero. However, in the strong-signal case, P_D is very close to unity (for $n > 10$), so that Eq. (5.3) does not really apply in the matched situation. What we really desire here is to compare the effects of detector law and the value of m for any two cases (m, m') of mismatch with $m, m' << n$. (This latter condition allows us to set $P_D = 1/2$.) From Eq. (5.3) we get at once

$$p_{oc} = (K_L \frac{n}{m^2})^{1/\nu}$$

$$p_{oc'} = (K_L \frac{n}{m'^2})^{1/\nu} \quad ,$$

and

$$\therefore \frac{p_{oc}}{p_{oc'}} = (\frac{m'}{m})^{2/\nu} \quad . \qquad (5.4)$$

Here, a much stronger dependence of p_{oc} on ν is evident, and with <u>small</u> ν $(<< 1)$, very large penalties of mismatch may be expected.

Figures 7 and 8 show p_{oc} for mismatch respectively for $\nu = 1.0$ and $\nu = 0.2$, based on Eqs. (5.3), where the condition p_o is large, or equivalently from Eq. (4.16) that $K_\nu \cdot n/m^2 >> 1$ holds. (The dotted lines are extrapolations outside the region of the condition.) Here P_D is again 1/2; Fig. 9 shows the needed $K_L(\nu)$, Eq. (5.3). Also plotted in Figs. 7 and 8 are the "exact" results for p_o vs sample-size n and $P_D = 1/2$, taken from Figs. 2, 3, and 4, enabling us to compare at once Cases (3) and (4). Thus, for instance, in Case (3) generally, the region of interest is for small n and small m (in mismatch), an' in Case (4), the region is for large n (and small m). Figures 10 and 11 contain essentially the same information as Figs. 7 and 8, but show now the penalty, i.e., (p_{oc}/p_o) in db, that mismatch incurs for the 50% probability of detection as a function of sample-size (n), for given m (and ν).

Generally speaking, we see that:

1. Decreasing the detector law (i.e., making $\nu < 1$) weakens performance very noticeably if ν is small [$0(10^{-1})$ or less]. The system fails completely when $\nu \to 0$, as then all signal information is destroyed by "super-clipping" [cf., remarks following Eq. (4.23)].
2. The penalties of mismatch are heavy in the strong-signal régime, and even more so in the weak-signal cases [cf., Eqs. (5.4) and (5.1)]. Spurious targets are accordingly more susceptible to misrepresentation in these latter cases.
3. Decreasing mismatch (i.e., increasing m), not unexpectedly reduces the mismatch penalty (cf., Figs. 6, 7, 8, 10, and 11).
4. As sample-size (n) is increased, larger inputs (p_{oc}) are required at mismatch ($m < n$) for a given probability of detection P_D (false-alarm probability α fixed), while at match ($m = n$) for this same value of P_D, smaller values of input p_o ($= p_{oc}$) are needed (cf., Figs. 7 and 8). This latter performance is the result of optimality, while the former reflects the sub-optimum character of mismatch.

These observations are not unexpected; their precise quantitative aspects, however, are.

Acknowledgment

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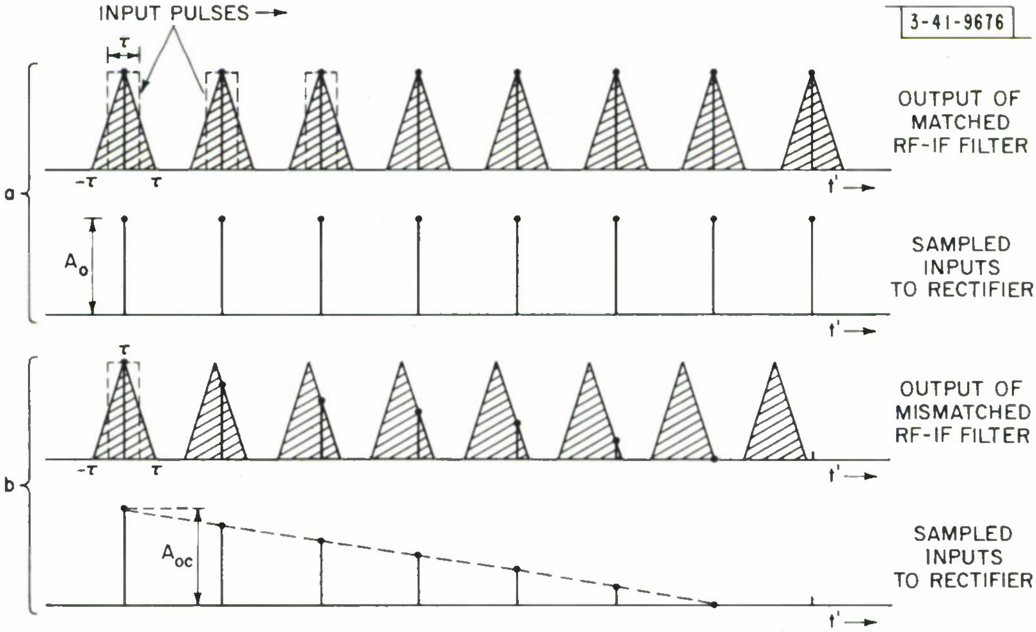

Fig. 1. (a) Output of matched filter at signal range (Doppler) gate with "1-point" samples to rectifier.

(b) Output of mismatched filter at gate with "1-point" samples to rectifier.

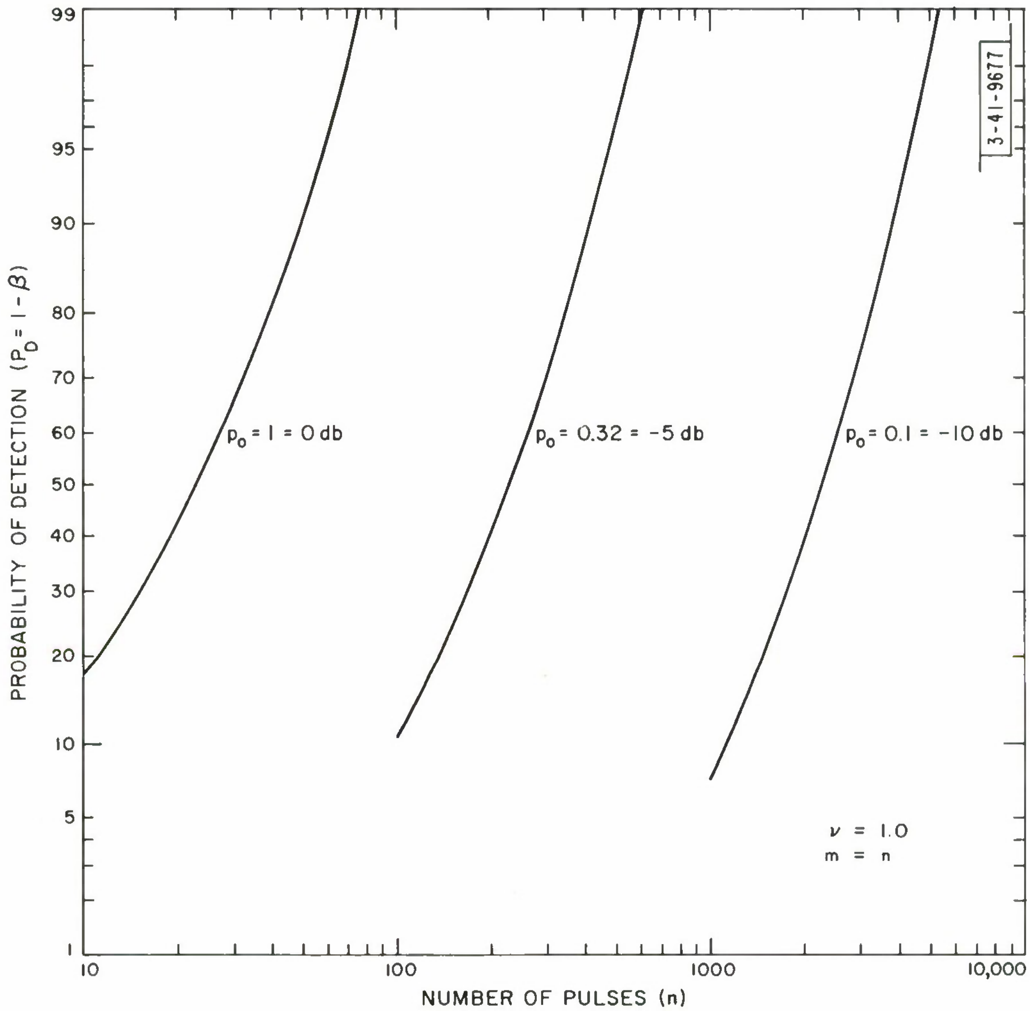

Fig. 2. Probability of detection (P_D) in the matched case vs number of pulses (n) for linear law ($\nu = 1.0$) detector with S/N per pulse (p_o) as parameter.

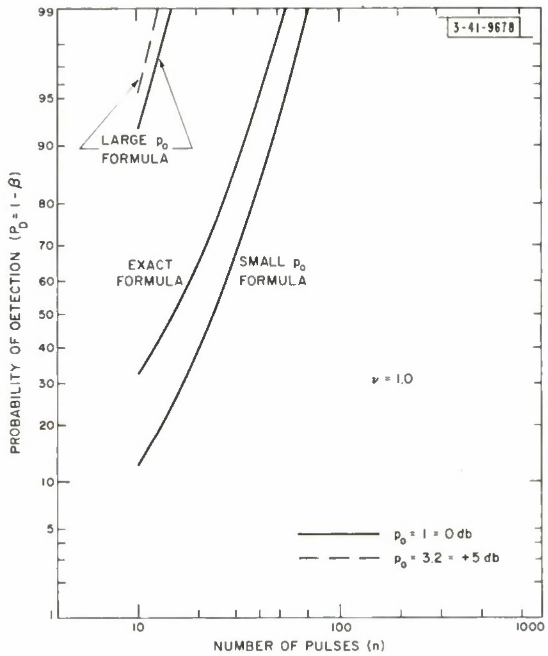

Fig. 3. Comparison of probability of detection in the matched case calculated from "exact method" and small-signal and large-signal approximations for linear detector (ν= 1.0).

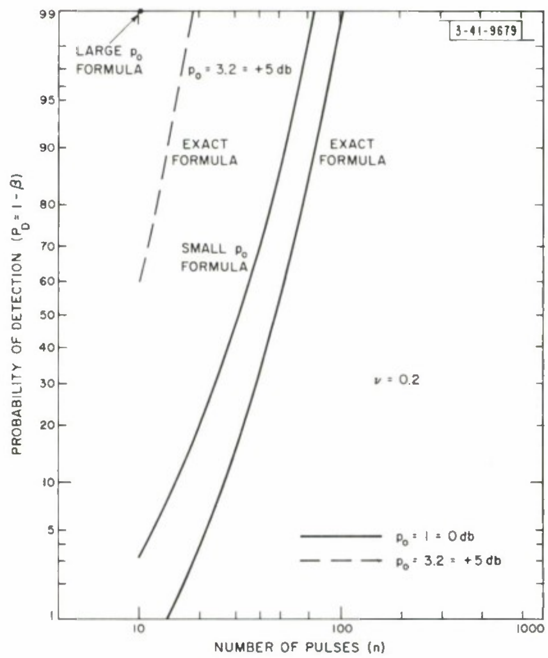

Fig. 4. Comparison of probability of detection in the matched case calculated from "exact method" and small-signal and large-signal approximations for detector law (ν = 0.2).

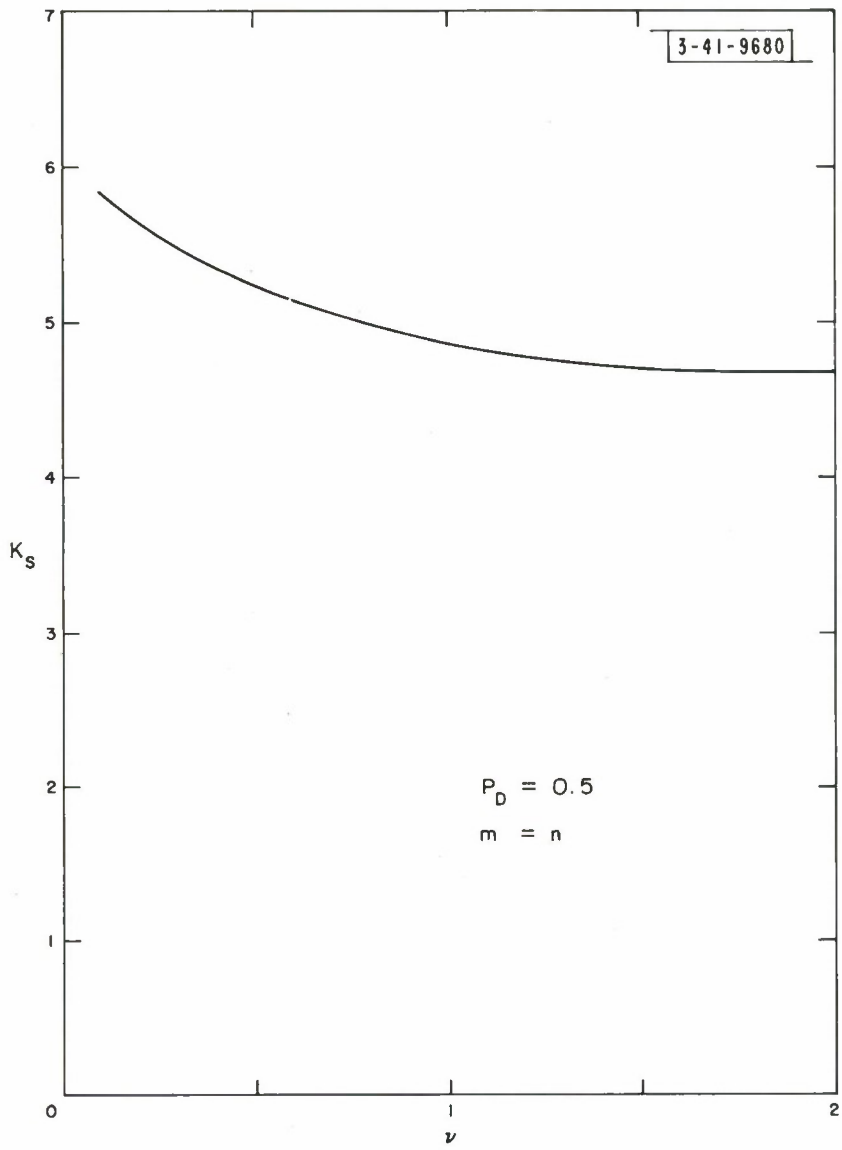

Fig. 5. $K_S(\nu)$ for probability of detection = 0.5 as a function of detector law (ν) in the matched, weak-signal régime.

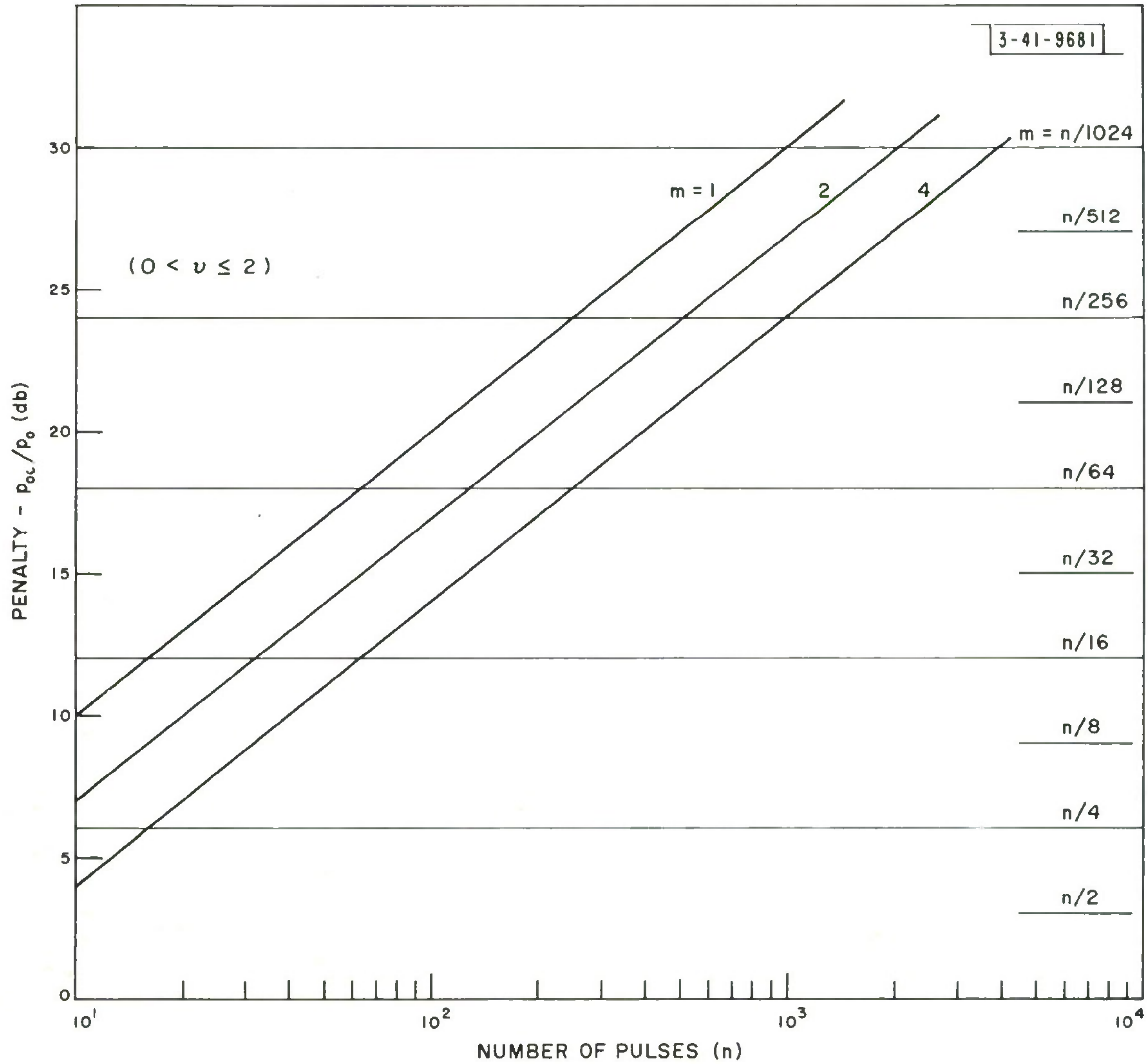

Fig. 6. Penalty, or increase in the S/N per pulse, as a result of velocity mismatch vs number of pulses for small-signal regime ($p_o << 1$).

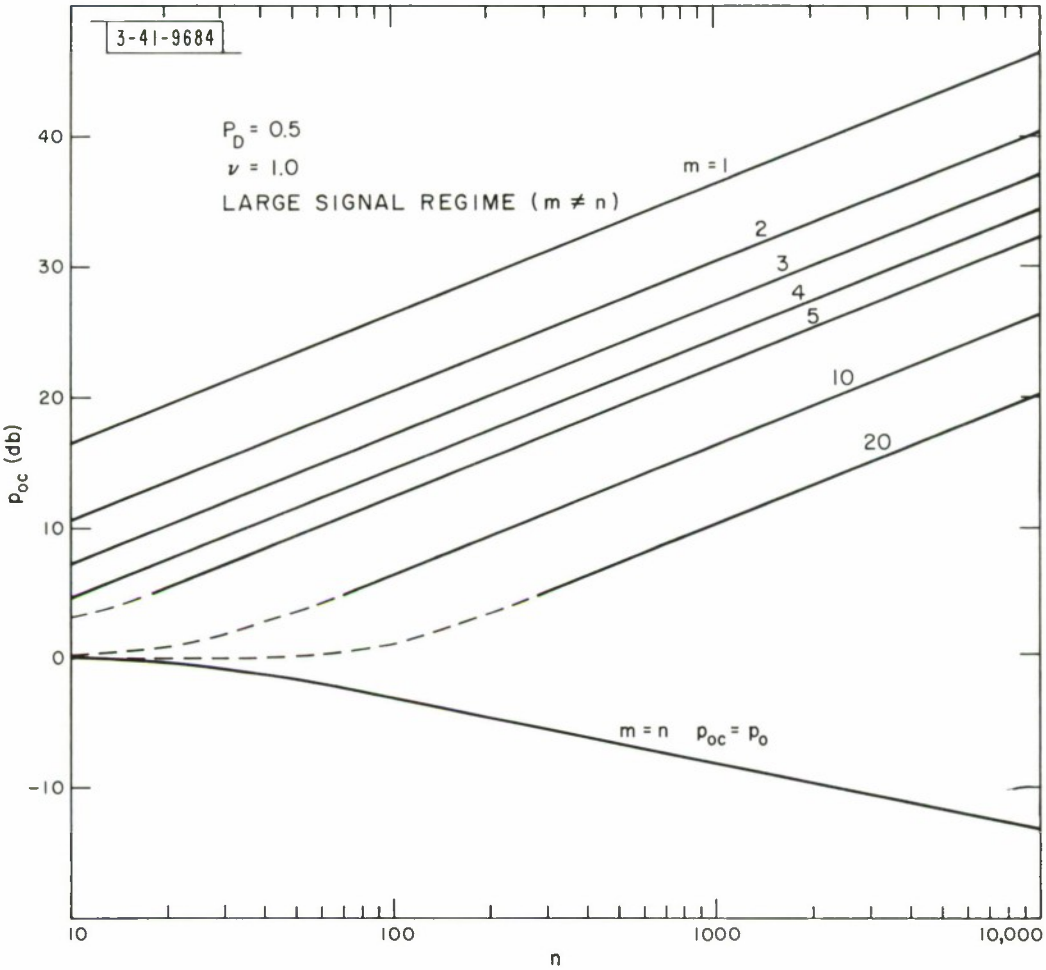

Fig. 7. Signal-to-noise ratio per pulse (p_o) required to achieve a probability of detection = 0.5 as a function of number of pulses for mismatch (m ≠ n) and match (m = n) cases.

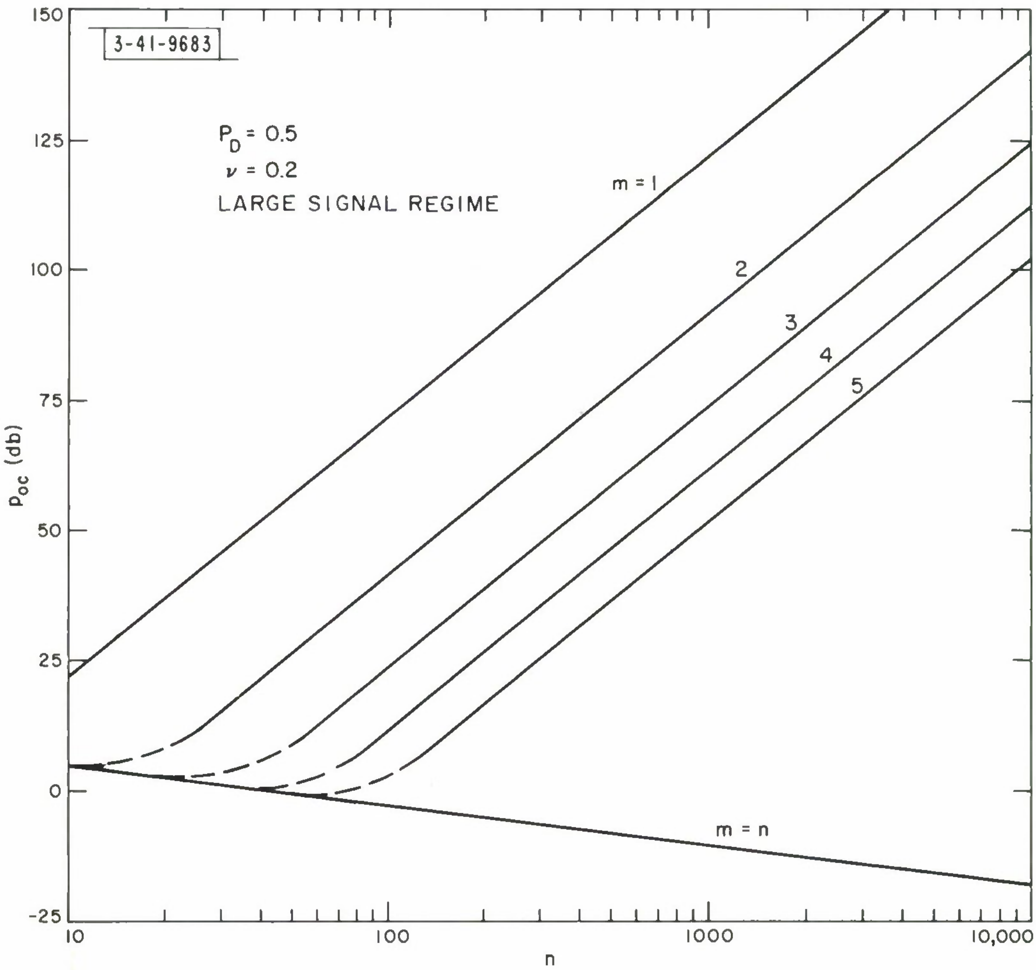

Fig. 8. Signal-to-noise ratio per pulse (p_o) required to achieve a probability of detection = 0.5 as a function of number of pulses for large-signal régime.

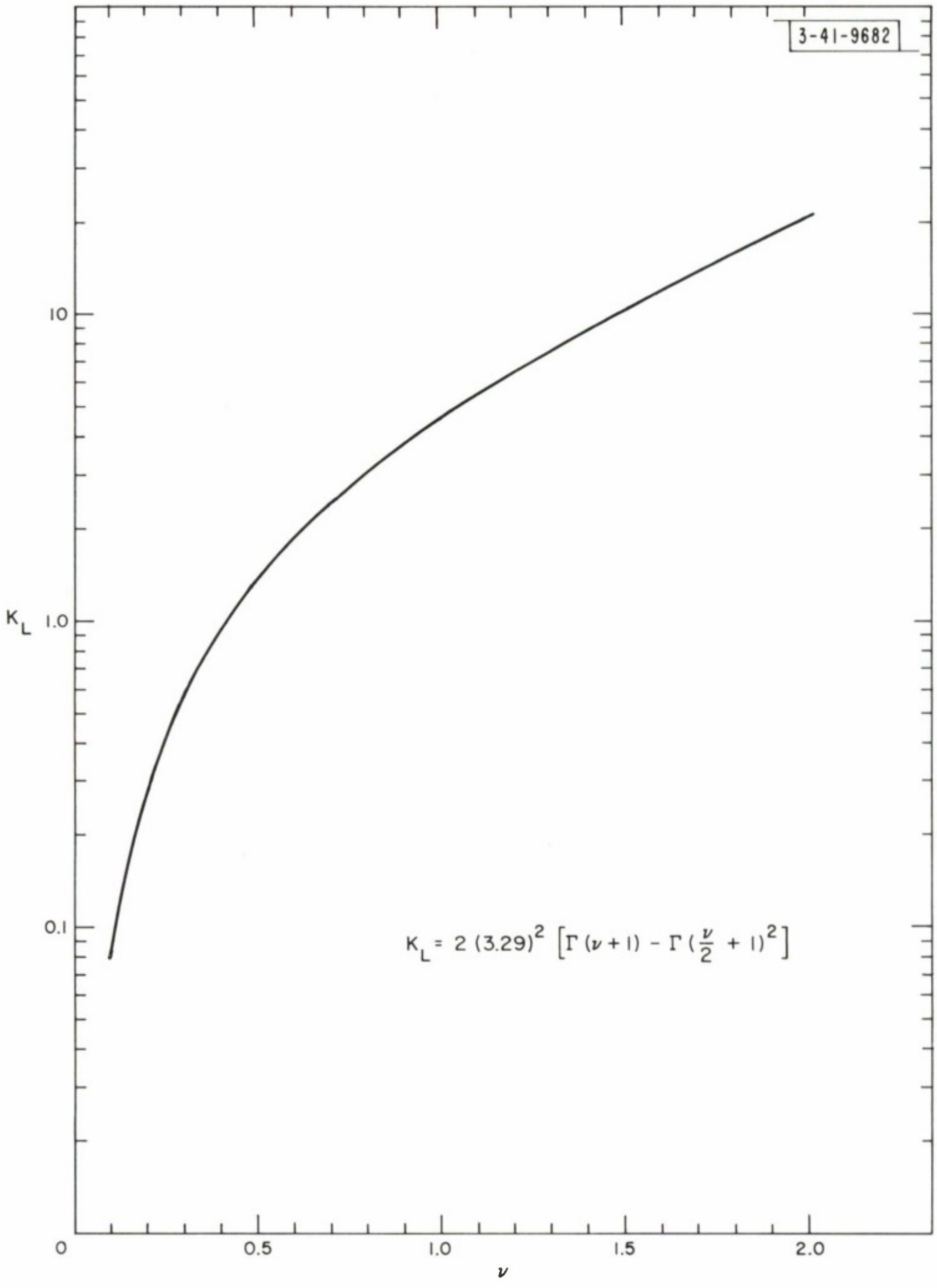

Fig. 9. $K_L(\nu)$ for probability of detection = 0.5 as a function of detector law (ν).

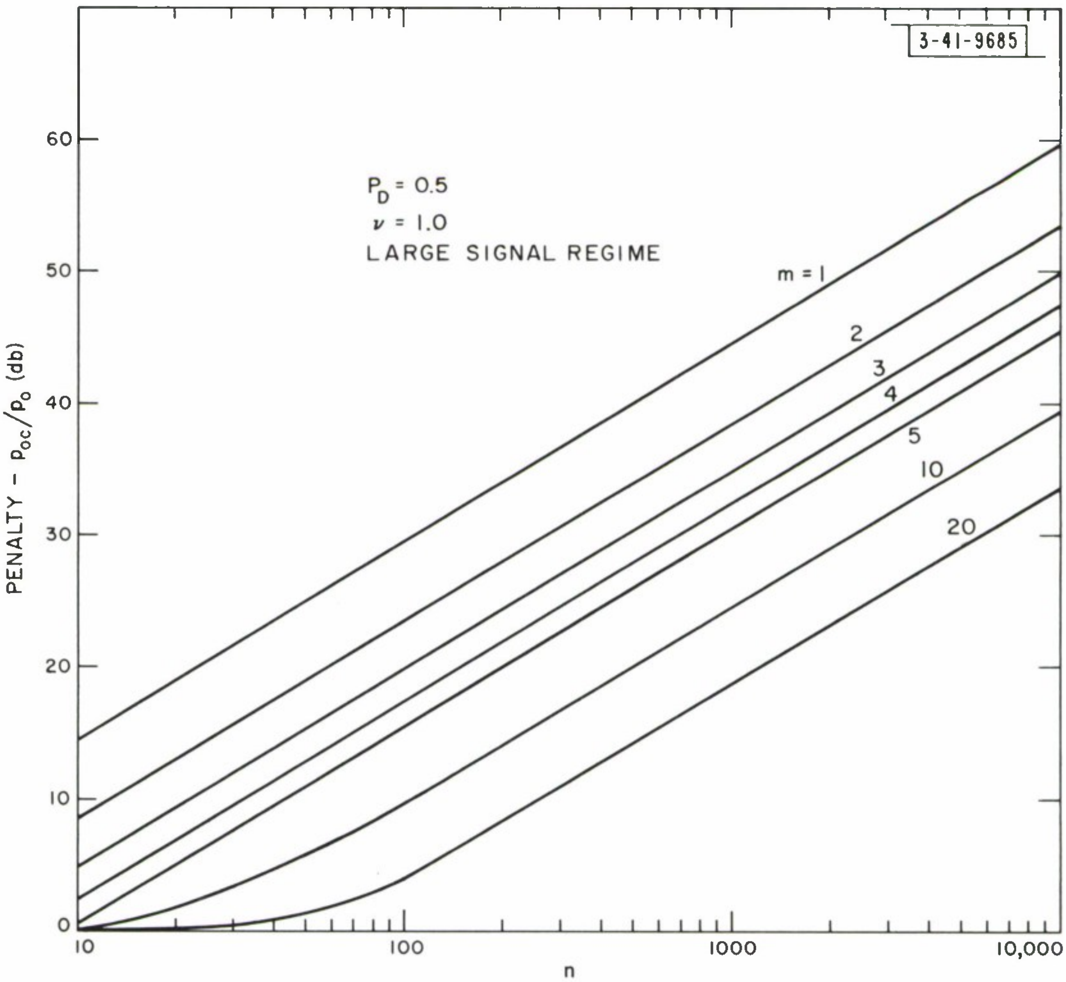

Fig. 10. Penalty, or increase in S/N ratio per pulse [Eq. (5.3)], as a result of velocity mismatch vs number of pulses for large-signal régime and $\nu = 1.0$.

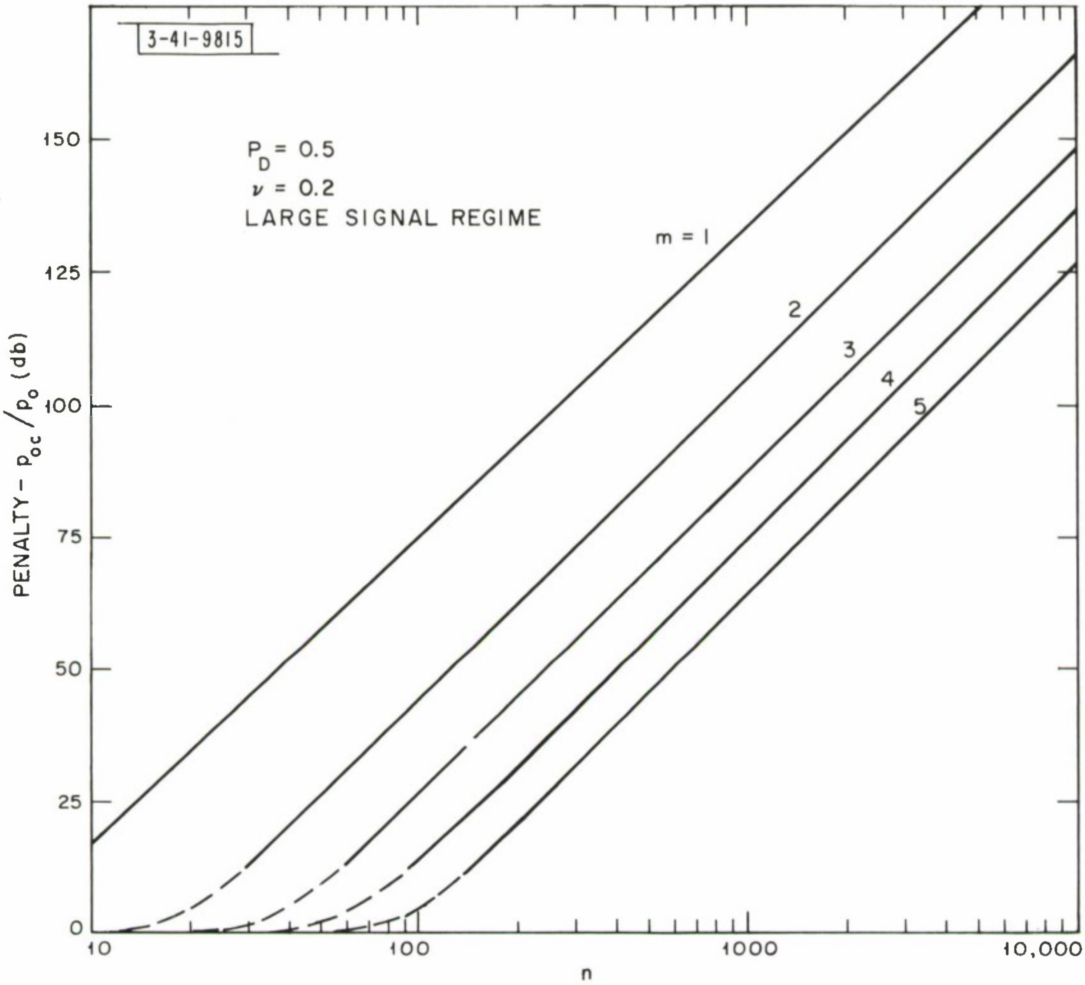

Fig. 11. Penalty, or increase in S/N ratio per pulse [Eq. (5.3)], as a result of velocity mismatch vs number of pulses for large-signal régime and $\nu = 0.2$.

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13. ABSTRACT

The probabilities of error and successful signal detection are determined for incoherent radar pulse trains observed by a suitably matched or mismatched receiver, which employs a half-wave νth-law rectifier before video presentation: single targets, normal noise, and incoherent reception are assumed. The effects of detector law, input signal strength, and sample size on performance are examined, and in particular, the modification of performance by mismatch is studied. Mismatch requires, not unexpectedly, a stronger input to achieve performance comparable to that at match. The amount and extent of these modifications are calculated for a variety of examples of interest. In all cases a (relatively) large-sample theory is required for the analysis when comparatively simple results are to be obtained. Exact expressions (all sample sizes) are found, however, in the quadratic case (ν = 2).

14. KEY WORDS

statistical analysis	radar targets	receivers
communication theory	detection	rectifiers
error probability		